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Histospline Projections on a Uniform Partition*

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We present a unified treatment of the periodic histospline projection of a function f on a uniform partition. We consider a given real number $v \in [0, 1]$ and obtain existence and uniqueness results for the n-degree periodic spline s determined by the values $\{\int_{x_i+vh}^{x_i+(v+1)h} s(x)\,dx\}_{i=0}^{N-1}$. For a function $f \in C_p^{n+1}[a, b]$ and a spline determined by the conditions $\int_{x_i+vh}^{x_i+(v+1)h} s(x)\,dx = \int_{x_i+vh}^{x_i+(v+1)h} f(x)\,dx$ $(i = 0, ..., N-1)$ we obtain error bounds of the form

$$\|f^{(k)} - s^{(k)}\|_\infty \simeq O(h^{n+1-k}) \qquad (k = 0, ..., n).$$

1. INTRODUCTION

Let $\Delta = \{x_i\}_{i=0}^N$ be a *uniform partition* of $[a, b]$ where $x_i = a + ih$ and $h = (b-a)/N$. An *n-degree spline* is a function $s \in C^{n-1}[a, b]$ such that s restricted to $[x_i, x_{i+1}]$ is a polynomial of degree at most n. It is a *periodic n-degree spline* if $s^{(k)}(a) = s^{(k)}(b)$ for all $k = 0, ..., n-1$.

This paper is devoted to the periodic n-degree histospline projection s determined by the condition

$$\int_{x_i+vh}^{x_i+(v+1)h} s(x)\,dx = \int_{x_i+vh}^{x_i+(v+1)h} f(x)\,dx$$

for all $i = 0, ..., N-1$ where $v \in [0, 1]$ is a given and fixed real number and f is a function with continuous and periodic derivatives through order $n+1$. This problem has been considered by several authors (see [1, 2, 7–12]). Using the linear dependence relationships which exist between any $n+1$ consecutive values of the definite integrals of the spline $\{\int_{x_i+vh}^{x_i+(v+1)h} s(x)\,dx\}_{i=0}^{n}$ and $n+2$ consecutive values of its kth derivative

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$\{s^{(k)}(x_i+uh)\}_{i=0}^{n+1}$ where $u, v \in [0, 1]$, we obtain existence and uniqueness results for the spline s and prove convergence results of the form $\|f^{(k)}-s^{(k)}\|_\infty \simeq O(h^{n+1-k})$ for all $k=0, ..., n$.

Throughout this paper we will use the following notation. The knots of the partition are $x_i = a+ih$ and for any $u \in \mathbb{R}$ $x_{i+u} = x_i + uh$. If g is a real valued function defined on the interval $[a, b]$, we will write $g_u = g(a+uh)$, $g^{(k)}$ is the kth derivative of g, $\mathrm{Var}(g)$ is the total variation of g on $[a, b]$, and $\|g\|_\infty$ is the uniform norm of g. $\|A\|_\infty$ is the uniform matrix norm. $A = \mathrm{circ}(a_1, ..., a_N)$ means that A is a (square) circulant matrix of order N with $a_1, ..., a_N$ on its first row (see [3, p. 66]). If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n and if $P = \mathrm{circ}(0, 1, 0, ..., 0)$ is of order $N \geqslant n+1$ then $p(P) = \mathrm{circ}(a_0, a_1, ..., a_n, 0, ..., 0)$. We also consider the following function spaces: $C^k[a, b]$, the space of functions with continuous derivatives through order k, $C_p^k[a, b]$, the space of functions $f \in C^k[a, b]$ such that $f^{(l)}(a) = f^{(l)}(b)$ for all $l = 0, ..., k$, and $\mathscr{P}_k$, the space of all polynomials of degree at most k.

2. Existence and Uniqueness Results

The existence result is based on the linear dependence relationships that relate the quantities $s_{i+u}^{(k)}$ and $\int_{x_{i+v}}^{x_{i+v+1}} s(x)\,dx$ ($k = 0, ..., n$ and $u, v \in [0, 1]$). These relationships, proved by Dubeau and Savoie [5], are

$$\frac{h^k}{(n+1)_{k+1}} \sum_{j=0}^{n+1} c_{n+1}^0(v, j)\, s_{l+j+u}^{(k)} = h^{-1} \sum_{j=0}^{n} c_n^k(u, j) \int_{x_{l+j+v}}^{x_{l+j+v+1}} s(x)\,dx \quad (1)$$

for all $k = 0, ..., n$ and $l \in \mathbb{Z}$ where $(n)_k = n!/(n-k)!$, $c_n^k(u, j) = (-1)^k \nabla^{n+1}(j+1-u)_+^{n-k}$, and ∇ is the backward difference operator.

Remark 1. In (1) and subsequent expressions, if $k = n$ we must consider right, or left, limits when u or v is 0, or 1.

Let us define the polynomials $p_n^k(u, z)$ as

$$p_n^k(u, z) = \sum_{j=0}^{n} c_n^k(u, j)\, z^j \quad (2)$$

for all $k = 0, ..., n$. If we consider (1) for $l = 0, ..., N-1$ and if we use the periodicity of s, we obtain

$$\frac{h^k}{(n+1)_{k+1}}\, p_{n+1}^0(v, P)\, s_{\Delta+u}^{(k)} = p_n^k(u, P) \int_0^1 s_{\Delta+v+w}\,dw, \quad (3)$$

where $s_{\Delta+u}^{(k)} = (s_u^{(k)}, s_{u+1}^{(k)}, ..., s_{N-1+u}^{(k)})$ and $P = \mathrm{circ}(0, 1, 0, ..., 0)$.

The polynomials $p_n^k(u, z)$ and the matrices $p_n^k(u, P)$ were studied in [6, 4] and we recall here, without proof, their main properties that will be used in this paper.

LEMMA 1. *$p_0^0(t, z) = 1$ and for $n \geqslant 1$*

(i) $p_n^0(1, z) = z p_n^0(0, z)$,

(ii) $p_n^k(u, z) = (z-1)^k p_{n-k}^0(u, z)$ *for* $k = 0, ..., n$,

(iii) $p_n^0(u, -1) = (-2)^n E_n(u)$,

where $E_n(\cdot)$ is the Euler polynomial of degree n.

LEMMA 2. *Let $P = \text{circ}(0, 1, 0, ..., 0)$ be of order $N \geqslant n+1$ and $n \geqslant 1$.*

(i) *If* $u \in [0, 1]$ *and* $\begin{cases} n \text{ is odd and } u \neq \frac{1}{2} \\ \text{or} \\ n \text{ is even and } u \neq 0 \text{ and } u \neq 1, \end{cases}$

then $p_n^0(u, P)$ is invertible and $\| p_n^0(u, P)^{-1} \|_\infty \leqslant 1/|p_n^0(u, -1)|$.

(ii) *If N is odd and* $\begin{cases} n \text{ is odd and } u = \frac{1}{2} \\ \text{or} \\ n \text{ is even and } u = 0 \text{ or } u = 1, \end{cases}$

then

(a) $p_n^k(u, z) = (z+1) q_n^k(u, z)$ *for* $k = 0, ..., n$ *where*

$$q_n^k(u, z) = \sum_{j=0}^{n-1} d_n^k(u, j) z^j \qquad \text{and} \qquad d_n^k(u, j) = \sum_{l=0}^{j} (-1)^{j-l} c_n^k(u, l),$$

(b) $q_n^0(u, -1) = -\frac{1}{2} p_{n+1}^0(u, -1)$,

(c) $p_n^0(u, P) = (P+I) q_n^0(u, P)$ *is invertible,* $(P+I)^{-1} = \frac{1}{2}\text{circ}(1, -1, ..., -1, 1)$, *and* $\|q_n^0(u, P)^{-1}\|_\infty \leqslant 1/|q_n^0(u, -1)|$.

We can now prove the following result.

THEOREM 1. *Let $N \geqslant n+2$ and $\Delta = \{x_i\}_{i=0}^N$ be a uniform partition of $[a, b]$ of step size h. If*

$$v \in [0, 1] \text{ and } \begin{cases} n \text{ is even and } v \neq \frac{1}{2} \\ \text{or} \\ n \text{ is odd and } v \neq 0 \text{ and } v \neq 1 \end{cases} \qquad \text{(Case A)}$$

or

$$N \text{ is odd and } \begin{cases} n \text{ is even and } v = \frac{1}{2} \\ \text{or} \\ n \text{ is odd and } v = 0 \text{ or } v = 1, \end{cases} \qquad \text{(Case B)}$$

then the periodic n-degree spline s is uniquely determined by the values $\{\int_{x_{i+v}}^{x_{i+v+1}} s(x)\, dx\}_{i=0}^{N-1}$. *Otherwise the spline does not exist or is not uniquely determined.*

Proof. From Lemma 2, the matrix $p_{n+1}^0(v, P)$ is invertible for $v \in [0, 1]$ if and only if we are in Case A or B. Since the inverse of a circulant matrix is a circulant matrix and two circulant matrices commute [3], it follows that

$$s_{\Delta+u}^{(k)} = \frac{(n+1)_{k+1}}{h^k} p_n^k(u, P)\, p_{n+1}^0(v, P)^{-1} \int_0^1 s_{\Delta+v+w}\, dw \qquad (4)$$

for all $u \in [0, 1]$ and $k = 0, ..., n$. To show that (4) defines a periodic n-degree spline, we consider the function $\tilde{s}$ defined as follows:

$$\tilde{s}(x) = (i+1)\text{th component of } s_{\Delta+u} \text{ if } x = x_i + uh.$$

The relation $p_n^0(1, P) = Pp_n^0(0, P)$ implies that $\tilde{s}$ is well defined and periodic on $[a, b]$. From the definition of $p_n^k(u, P)$ we obtain

$$\tilde{s}^{(k)}(x) = (i+1)\text{th component of } s_{\Delta+u}^{(k)} \text{ if } x = x_i + uh$$

and $\tilde{s}^{(k)}$ is a piecewise polynomial of degree $n-k$ for $k = 0, ..., n$. Using the relation $p_n^k(1, P) = Pp_n^k(0, P)$, we show that $\tilde{s}^{(k)}$ is continuous and periodic for $k = 0, ..., n-1$. ∎

3. Derivation of Error Bounds

We consider a given periodic function $f \in C_p^{n+1}[a, b]$ and the periodic n-degree spline s define over a uniform partition $\Delta = \{x_i\}_{i=0}^N$ of $[a, b]$ by the relations

$$\int_{x_{i+v}}^{x_{i+v+1}} s(x)\, dx = \int_{x_{i+v}}^{x_{i+v+1}} f(x)\, dx$$

for all $i = 0, ..., N-1$ and where $v \in [0, 1]$.

Using (3), the *remainder function* $e=f-s$ satisfies the equation

$$p_{n+1}^{0}(v, P)\, e_{\Delta+u}^{(k)} = p_{n+1}^{0}(v, P)\, f_{\Delta+u}^{(k)} - \frac{(n+1)_{k+1}}{h^k}\, p_n^k(u, P) \int_0^1 f_{\Delta+v+w}\, dw \qquad (5)$$

for $k=0, ..., n$. Each line of the right-hand side of the system (5) is of the form

$$L_n^k(u, v; g) = \sum_{j=0}^{n+1} c_{n+1}^{0}(v, j)\, g_{j+u}^{(k)} - \frac{(n+1)_{k+1}}{h^k} \sum_{j=0}^{n} c_n^k(u, j) \int_0^1 g_{j+v+w}\, dw,$$

where $g=T_l f$ with T_l the shift operator defined by $T_l f(x) = f(x+lh)$. Since $L_n^k(u, v; \cdot)$ is a linear functional that vanishes for all $p \in \mathscr{P}_n$, it follows from the Peano Kernel Theorem that

$$L_n^k(u, v; g) = \int_a^b K_n^k(u, v; t)\, g^{(n+1)}(t)\, dt \qquad (6)$$

for all $g \in C_p^{n+1}[a, b]$ where

$$K_n^k(u, v; t) = \frac{1}{n!} L_{n,x}^k(u, v; (x-t)_+^n)$$

and $L_{n,x}^k(u, v; (x-t)_+^n)$ means that the functional $L_n^k(u, v; \cdot)$ is applied to $(x-t)_+^n$ considered as a function of x.

Using the shift operator and the change of variable $t = a + \theta h$, (6) becomes

$$K_n^k(u, v; T_l f) = h^{n+1-k} \int_0^{n+2} \bar{K}_n^k(u, v; \theta)\, f^{(n+1)}(x_l + \theta h)\, d\theta,$$

where

$$\bar{K}_n^k(u, v; \theta) = \frac{1}{(n-k)!} \left[\sum_{j=0}^{n+1} c_{n+1}^{0}(v, j)(j+u-\theta)_+^{n-k} - (n+1) \sum_{j=0}^{n} c_n^k(u, j) \int_0^1 (j+v+w-\theta)_+^n\, dw \right].$$

We can now write (5) as

$$p_{n+1}^{0}(v, P)\, e_{\Delta+u}^{(k)} = h^{n+1-k} \int_0^{n+2} \bar{K}_n^k(u, v; \theta)\, f_{\Delta+\theta}^{(n+1)}\, d\theta \qquad (7)$$

and prove the following results.

THEOREM 2 (Case A). *Let $N \geqslant n+2$, $\Delta = \{x_i\}_{i=0}^{N}$ be a uniform partition of $[a, b]$ of step size h, and $f \in C_p^{n+1}[a, b]$. If*

$$v \in [0, 1] \text{ and } \begin{cases} n \text{ is even and } v \neq \frac{1}{2} \\ \text{or} \\ n \text{ is odd and } v \neq 0 \text{ and } v \neq 1, \end{cases}$$

then there exist constants $C_n^k(v)$, independent of the partition, such that

$$\|f^{(k)} - s^{(k)}\|_\infty \leqslant C_n^k(v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty.$$

Moreover,

$$C_n^k(v) = \frac{1}{2^{n+1}\,|E_{n+1}(v)|} \sup_{u \in [0,1]} \int_0^{n+2} |\bar{K}_n^k(u, v; \theta)|\, d\theta,$$

where $E_n(\cdot)$ is the Euler polynomial of degree n.

Proof. From (7) it follows that

$$\|e_{\Delta+u}^{(k)}\|_\infty \leqslant h^{n+1-k}\, \|f^{(n+1)}\|_\infty\, \|p_{n+1}^0(v, P)^{-1}\|_\infty \int_0^{n+2} |\bar{K}_n^k(u, v; \theta)|\, d\theta$$

for all $u \in [0, 1]$. The result follows from the relation

$$\|e^{(k)}\|_\infty = \sup_{u \in [0,1]} \|e_{\Delta+u}^{(k)}\|_\infty \tag{8}$$

and Lemmas 1 and 2. ∎

THEOREM 3 (Case B). *Let $N \geqslant n+2$, $\Delta = \{x_i\}_{i=0}^{N}$ be a uniform partition of $[a, b]$ of step size h, $f \in C_p^{n+1}[a, b]$, and $f^{(n+1)}$ be of bounded variation. If*

$$N \text{ is odd and } \begin{cases} n \text{ is even and } v = \frac{1}{2} \\ \text{or} \\ n \text{ is odd and } v = 0 \text{ or } v = 1, \end{cases}$$

then there exist constants $\bar{C}_n^k(v)$, independent of the partition, such that

$$\|f^{(k)} - s^{(k)}\|_\infty \leqslant \bar{C}_n^k(v)\, h^{n+1-k}\, [\|f^{(n+1)}\|_\infty + \mathrm{Var}(f^{(n+1)})].$$

Moreover,

$$\bar{C}_n^k(v) = \frac{1}{2^{n+2}\,|E_{n+2}(v)|} \sup_{u \in [0,1]} \int_0^{n+2} |\bar{K}_n^k(u, v; \theta)|\, d\theta.$$

Proof. From Lemma 2, we have $p^0_{n+1}(v, P) = (P+I)\, q^0_{n+1}(v, P)$. It follows from (7) that

$$(I+P)\, e^{(k)}_{\Delta+u} = h^{n+1-k} q^0_{n+1}(v, P)^{-1} \int_0^{n+2} \bar{K}^k_n(u, v; \theta)\, f^{(n+1)}_{\Delta+\theta}\, d\theta$$

and

$$\|(I+P)\, e^{(k)}_{\Delta+u}\|_\infty \leqslant h^{n+1-k}\, \|f^{(n+1)}\|_\infty\, \|q^0_{n+1}(v, P)^{-1}\|_\infty \int_0^{n+2} |\bar{K}^k_n(u, v; \theta)|\, d\theta.$$

For N odd $I+P$ is invertible and we have

$$(I-P)\, e^{(k)}_{\Delta+u} = h^{n+1-k} q^0_{n+1}(v, P)^{-1} \int_0^{n+2} \bar{K}^k_n(u, v; \theta)(I+P)^{-1}(I-P)\, f^{(n+1)}_{\Delta+\theta}\, d\theta$$

since circulant matrices commute. But since $(I+P)^{-1} = \frac{1}{2}\text{circ}(1, -1, ..., -1, 1)$ and $(I+P)^{-1}(I-P) = \text{circ}(0, -1, 1, ..., -1, 1)$, then

$$\|(I+P)^{-1}(I-P)\, f^{(n+1)}_{\Delta+\theta}\|_\infty \leqslant \text{Var}(f^{(n+1)})$$

and we obtain

$$\|(I-P)\, e^{(k)}_{\Delta+u}\|_\infty \leqslant h^{n+1-k}\, \text{Var}(f^{(n+1)})\|\, q^0_{n+1}(v, P)^{-1}\|_\infty \times \int_0^{n+2} |\bar{K}^k_n(u, v; \theta)|\, d\theta.$$

The result follows from (8), the identity

$$e^{(k)}_{\Delta+u} = \tfrac{1}{2}(I+P)\, e^{(k)}_{\Delta+u} + \tfrac{1}{2}(I-P)\, e^{(k)}_{\Delta+u}, \tag{9}$$

and Lemmas 1 and 2. ∎

Finally, in Case B there exist values of k and u for which the bounds of $\|f^{(k)}_{\Delta+u} - s^{(k)}_{\Delta+u}\|_\infty$ are independent of the variation of $f^{(n+1)}$ as in Case A.

Consider Case B and assume that

$$n-k \geqslant 1 \text{ and } \begin{cases} n-k \text{ is odd and } u = \frac{1}{2} \\ \text{or} \\ n-k \text{ is even and } u = 0 \text{ or } u = 1, \end{cases}$$

then it can be shown (see [5, Corollary 8]) that the n-degree spline satisfies

$$\frac{h^k}{(n+1)_{k+1}} \sum_{j=0}^{n} d_{n+1}^0(v, j)\, s_{l+j+u}^{(k)} = h^{-1} \sum_{j=0}^{n-1} d_n^k(u, j) \int_{x_{l+j+v}}^{x_{l+j+v+1}} s(x)\, dx.$$

Using the periodicity of s, we obtain

$$\frac{h^k}{(n+1)_{k+1}}\, q_{n+1}^0(v, P)\, s_{\Delta+u}^{(k)} = q_n^k(u, P) \int_0^1 s_{\Delta+v+w}\, dw \tag{10}$$

which is nothing but (3) where we have cancelled the factor $I+P$ on both sides.

In this case, following Theorem 2, we have the following result.

THEOREM 4. *Let $N \geqslant n+2$, $\Delta = \{x_i\}_{i=0}^N$ be a uniform partition of $[a, b]$ of step size h, and $f \in C_p^{n+1}[a, b]$. If*

$$N \text{ is odd and } \begin{cases} n \text{ is even and } v = \frac{1}{2} \\ \text{or} \\ n \text{ is odd and } v = 0 \text{ or } v = 1 \end{cases}$$

and if

$$n-k \geqslant 1 \text{ and } \begin{cases} n-k \text{ is odd and } u = \frac{1}{2} \\ \text{or} \\ n-k \text{ is even and } u = 0 \text{ or } u = 1, \end{cases}$$

then there exist constants $\hat{C}_n^k(u, v)$, independent of the partition, such that

$$\|f_{\Delta+u}^{(k)} - s_{\Delta+u}^{(k)}\|_\infty \leqslant \hat{C}_n^k(u, v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty.$$

Moreover,

$$\hat{C}_n^k(u, v) = \frac{1}{2^{n+1}\, |E_{n+2}(v)|} \int_0^{n+1} |\hat{K}_n^k(u, v; \theta)|\, d\theta,$$

where

$$\hat{K}_n^k(u, v; \theta) = \frac{1}{(n-k!)} \left[\sum_{j=0}^{n} d_n^0(v, j)(j+u-\theta)_+^{n-k} \right.$$
$$\left. -(n+1) \sum_{j=0}^{n-1} d_n^k(u, j) \int_0^1 (j+v+w-\theta)_+^n\, d\theta \right].$$

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